Quantum kinetic equation in weak turbulence

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The quantum kinetic equation used in the study of weak turbulence is reconsidered in the context of a theory with a generic quartic interaction. The expectation value of the time derivative of the mode number operators is computed in a perturbation expansion that places the large diagonal component of the quartic term in the unperturbed Hamiltonian. Although one is not perturbing around a free field theory, the calculation is easily tractable owing to the fact that the unperturbed Hamiltonian can be written solely in terms of the mode number operators.

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I. INTRODUCTION

In one approach to the statistical description of weak turbulence, a central role is played by the kinetic wave equation [1, 2]. This equation for the time derivative of the mode numbers has been derived for both classical and quantum systems in a perturbation series by expanding about a free field (harmonic oscillator) theory. In this paper, we will reconsider this derivation for a quantum mechanical system whose Hamiltonian is a sum of generic quadratic and quartic terms. Our perturbation expansion will perturb around an operator that contains the diagonal component of the quartic term together with the usual quadratic, or free field, component. Since one is interested in the expectation values of fields between states with large mode numbers, it is sensible to include as much of these in the unperturbed Hamiltonian as the calculation permits. We do not need to assume that the coupling to all the quartic terms in the Hamiltonian is small; the diagonal part can be arbitrarily large in our approach.

We begin in the following section with a precise statement of the theory under consideration and an encapsulation of the interaction picture used to carry out the derivation of the quantum kinetic equation. At this order, the largest terms are cubic in the mode numbers and these we calculate explicitly. We will find some additional terms not present in another derivation of the quantum kinetic equation [1] and then go on to consider under what conditions one might expect those corrections to be small. A discussion of stationary solutions to the kinetic equation then follows.

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II. THE QUANTUM KINETIC EQUATION

Let us consider a quantum mechanical system based on the Hamiltonian

$$H = \sum_k \omega_k a_k^\dagger a_k + \sum_{k_1,\ldots,k_4} T_{k_1 k_2 k_3 k_4} a_{k_1}^\dagger a_{k_2}^\dagger a_{k_3} a_{k_4}, \quad (1)$$

which contains generic quadratic and quartic terms; the number d of spatial dimensions in which the system evolves is arbitrary. The free field oscillator energies ω_k are assumed given and constitute part of the specification of the system. The function $T_{k_1 k_2 k_3 k_4}$ includes, by definition, the momentum conserving factor $\delta_{k_1+k_2,k_3+k_4}$ (both the vector k_i and the δ function should be understood as d-dimensional quantities). Beyond the implicit symmetry properties, which are that $T_{k_1 k_2 k_3 k_4}$ is symmetric in the first two and last two indices and that under complex conjugation $T^*_{k_1 k_2 k_3 k_4} = T_{k_3 k_4 k_1 k_2}$, this coefficient may contain further momentum dependence, which we will otherwise not restrict in the derivation of the kinetic equation. As usual, $a_k^\dagger$ and a_k in Eq. (1) denote Bose creation and annihilation operators and obey the commutation relation

$$[a_k, a_l^\dagger] = \delta_{k,l} \ . \quad (2)$$

It is convenient to use the number operator $\hat{n}_k = a_k^\dagger a_k$ for each mode in our system. The states that diagonalize these number operators satisfy [3] $\hat{n}_k|n_k\rangle = n_k|n_k\rangle$, $a_k|n_k\rangle = \sqrt{n_k}\,|n_k - 1\rangle$, and $a_k^\dagger|n_k\rangle = \sqrt{n_k + 1}\,|n_k + 1\rangle$ and are clearly labeled by their eigenvalues.

A perturbation series (see [3] for a thorough exposition) begins by splitting the Hamiltonian into an unperturbed component H_0 and a "small" component H_1. In our approach, we will place the diagonal of the quartic component in the unperturbed sector, so that $H = H_0 + H_1$, with

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$$H_0 = \sum_k (\omega_k - 2T_k)\,\hat{n}_k + 2\sum_{k,l} T_{kl}\,\hat{n}_k\,\hat{n}_l\ , \tag{3}$$

$$H_1 = \sum_{k_1,\ldots,k_4} T'_{k_1k_2k_3k_4}\,a^\dagger_{k_1}\,a^\dagger_{k_2}\,a_{k_3}\,a_{k_4}$$

and where we have introduced the notation $T_k = T_{kkkk}$, $T_{kl} = T_{klkl}$, and

$$T'_{k_1k_2k_3k_4} = \begin{cases} T_{k_1k_2k_3k_4} & \text{if } k_1 \neq k_3 \text{ or } k_4 \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

We can express this equivalently as

$$T'_{k_1k_2k_3k_4} = \Big(1 - \delta_{k_1k_3}\,\delta_{k_2k_4} - \delta_{k_1k_4}\,\delta_{k_2k_3} + \delta_{k_1k_2}\,\delta_{k_1k_3}\,\delta_{k_2k_4}\Big)\,T_{k_1k_2k_3k_4}\ . \tag{5}$$

It is important to emphasize that we need not assume that the coefficient functions T_{kl} be small, as they are part of the unperturbed Hamiltonian. The validity of the perturbation expansion depends, however on $T'_{k_1k_2k_3k_4}$ being small relative to the mode numbers.

In the Heisenberg representation of quantum mechanics, operators are time dependent while the states are time independent. Given some operator A, its time evolution as a Heisenberg operator $A_H(t)$ satisfies

$$\frac{d}{dt}A_H(t) = i\,[H, A_H(t)] \tag{6}$$

and one may equivalently write this as $A_H(t) = \exp[iHt]\,A_H(0)\,\exp[-iHt]$. Any expectation value in this representation therefore satisfies

$$\frac{d}{dt}\,\langle\Psi_1|A_H(t)|\Psi_2\rangle = \left\langle\Psi_1\right|\frac{d}{dt}A_H(t)\left|\Psi_2\right\rangle = \langle\Psi_1|\,i[H, A_H(t)]\,|\Psi_2\rangle\ , \tag{7}$$

since the states $|\Psi_a\rangle$ are independent of t. We are interested in the case $A = \hat{n}_k$ for large t and we will compute

$$\lim_{t\to\infty}\,\langle\Psi|\,i[H, \hat{n}_{kH}(t)]\,|\Psi\rangle \tag{8}$$

for some state $|\Psi\rangle$, which we will later specify. This is the precise meaning we ascribe to the time derivative of the mode number appearing in other presentations [1, 2] of the kinetic wave equation.

For the purposes of perturbation theory, one moves to the interaction picture where the following relations hold:

$$\langle\Psi_1|\mathcal{O}_H(t)|\Psi_2\rangle = \langle\Psi_1(t)|\mathcal{O}_I(t)|\Psi_2(t)\rangle, \tag{9}$$

$$|\Psi_a(t)\rangle = \exp[i\,H_0\,t]\,\exp[-i\,H\,(t-t')] \times \exp[-i\,H_0\,t']|\Psi_a(t')\rangle\ ,$$

$$\mathcal{O}_I(t) = \exp[i\,H_0\,(t-t')]\mathcal{O}_I(t') \times \exp[-i\,H_0\,(t-t')]$$

for all operators $\mathcal{O}$ and all states $|\Psi_a\rangle$. Interestingly, the time evolution of the operators $a^\dagger_{kI}$ and a_{kI} in this model is very simple in spite of the fact that they do not evolve via a free field Hamiltonian. It is not difficult to first show that

$$[a_k, H_0] = \left(\omega_k + 4\sum_l T_{kl}\,\hat{n}_l\right)\,a_k\ , \tag{10}$$

and using this one can quickly prove

$$a_{kI}(t) = \exp[i\,H_0\,t]\,a_{kI}(0)\,\exp[-i\,H_0\,t] \tag{11}$$

$$= \exp\left[-i\,t\,\left(\omega_k + 4\sum_l T_{kl}\,\hat{n}_l\right)\right]\,a_{kI}(0)$$

$$= a_{kI}(0)\,\exp\left[-i\,t\,\left(\omega_k - 4T_k + 4\sum_l T_{kl}\,\hat{n}_l\right)\right]\ .$$

The combination of operators in the second line of (9) is conveniently denoted by

$$U(t,t') = \exp[i\,H_0\,t]\,\exp[-i\,H\,(t-t')]\,\exp[-i\,H_0\,t'] = 1 - i\int_{t'}^{t} d\tau H_{1I}(\tau)\,U(\tau,t'). \tag{12}$$

To lowest order in the interaction H_1, we just set $U(\tau,t') = 1$ on the right hand side of Eq. (12). Our goal is to compute

$$\left\langle\frac{d}{dt}\hat{n}_k\right\rangle \equiv \lim_{t\to\infty}\,\langle\Psi(-t)|\,U^\dagger(t,-t) \times\mathcal{O}_I(t)\,U(t,-t)\,|\Psi(-t)\rangle\ , \tag{13}$$

where $\mathcal{O} = i[H_1, \hat{n}_k]$. We should emphasize that the states on both sides of this expectation value are *in* states in the sense of scattering theory as both are at $-\infty$.

Noticing that H_0 commutes with $\hat{n}_k$, the computation of Eq. (13) to the lowest order in perturbation theory reduces to

$$\lim_{t\to\infty}\left\langle\Psi(-t)\right|i[H_{1I}(t), \hat{n}_k] + \int_{-t}^{t}\,[[H_{1I}(t), \hat{n}_k], H_{1I}(\tau)]\,d\tau\left|\Psi(-t)\right\rangle\ . \tag{14}$$

We assume that the state $|\Psi(-\infty)\rangle$ is an eigenstate of the number operators n_k. It is not difficult to show that the first order term in (14) does not contribute; to see this one simply computes

$$[H_1, \hat{n}_k] = 2\sum_{k_1,k_2,k_3}(T'_{k_1,k_2,k_3k}\,a^\dagger_{k_1}\,a^\dagger_{k_2}\,a_{k_3}\,a_k - T'_{k_1kk_2k_3}\,a^\dagger_{k_1}\,a^\dagger_k\,a_{k_2}\,a_{k_3})\ . \tag{15}$$

When we take this between the same states, there is no way to pair the creation and annihilation operators as T' is off diagonal and hence the first term in (14) manifestly vanishes. However, we should point out that this term would also vanish even if we had not subtracted out the diagonal and had instead made the normal perturbative expansion around the quadratic term in H; the cancellation would then involve a mixing of both terms in Eq. (15).

One of the ingredients needed in this computation is

the expectation value

$$\langle a^\dagger_{k_1}\, a^\dagger_{k_2}\, a_{k_3}\, a_{k_4}\, a^\dagger_{l_1}\, a^\dagger_{l_2}\, a_{l_3}\, a_{l_4}\rangle \ , \tag{16}$$

where k_1 or $k_2 \neq k_3$ or k_4, and l_1 or $l_2 \neq l_3$ or l_4. The states on both sides are identical and are eigenstates of all the number operators. A simple computation yields

$$\begin{aligned}&(n_{l_1}+1)\, n_{l_3}[(\delta_{l_1 k_3}\,\delta_{l_2 k_4}+\delta_{l_1 k_4}\,\delta_{l_2 k_3})\ (n_{l_2}+1)\\ &\quad -\delta_{l_1 l_2}\,\delta_{l_1 k_3}\,\delta_{l_2 k_4}\,n_{l_2}]\\ &\quad \times[(\delta_{l_3 k_1}\,\delta_{l_4 k_2}+\delta_{l_3 k_2}\,\delta_{l_4 k_1})\,n_{l_4}\\ &\quad -\delta_{l_3 l_4}\,\delta_{l_3 k_1}\,\delta_{l_4 k_2}\,(n_{l_4}+1)]\ .\end{aligned} \tag{17}$$

We should note that it is not sufficient for the purposes of our calculation, even working in the large-n_l limit, to keep only the most dominant terms in the above expression, which are fourth order in these mode numbers. When this expression is used in our calculation, we will see that the fourth-order terms cancel and the next to leading-order terms remain. For this reason we have been careful to take account of the possibility that l_1 equals l_2, and so on, in this expectation value; no assumptions (e.g., random phase approximation; see [1]) have been made in obtaining the expression in (17).

It is straightforward, though rather tedious, to assemble the above pieces, and the tree level expression for the quantity in (13) is found to be

$$\begin{aligned}\left\langle \frac{d}{dt}\hat{n}_k\right\rangle = 8\,\pi \sum_{k_1,k_2,k_3} &|T'_{k_1 k_2 k_3 k}|^2\ \delta(k_1,k_2,k_3,k)\\ &\times[s_3(k_1,k_2,k_3,k)+s_2(k_1,k_2,k_3,k)\\ &+s_1(k_1,k_2,k_3,k)\ ]\ ,\end{aligned} \tag{18}$$

with the functions $s_a(k_1,k_2,k_3,k)$ given by

$$\begin{aligned}s_3 = 4\ &(n_{k_1}\,n_{k_2}\,n_{k_3}+n_{k_1}\,n_{k_2}\,n_k\\ &-n_{k_1}\,n_{k_3}\,n_k-n_{k_2}\,n_{k_3}\,n_k)\\ &-2\,\delta_{k_1 k_2}\,(n_{k_1}\,n_{k_2}\,n_{k_3}+n_{k_1}\,n_{k_2}\,n_k)\\ &+2\,\delta_{k_3 k}\,(n_{k_1}\,n_{k_3}\,n_k+n_{k_2}\,n_{k_3}\,n_k)\ ,\end{aligned} \tag{19}$$

$$\begin{aligned}s_2 = 4\ &(n_{k_1}\,n_{k_2}-n_{k_3}\,n_k\ )\\ &-2\,\delta_{k_1 k_2}\,(n_{k_1}\,n_{k_2}+n_{k_1}\,n_k+n_{k_1}\,n_{k_3})\\ &+2\,\delta_{k_3 k}\,(n_{k_1}\,n_{k_3}+n_{k_3}\,n_k+n_{k_2}\,n_{k_3})\ ,\end{aligned}$$

$$s_1 = -2\,\delta_{k_1 k_2}\,n_{k_1}+2\,\delta_{k_3 k}\,n_{k_3}\ .$$

For large values of the mode numbers, the s_3 term that is cubic in those variables will dominate. The energy conserving delta function $\delta(k_1,k_2,k_3,k)$, which arose from an integration over τ, is given by

$$\begin{aligned}\delta\Big(&\omega_{k_1}+\omega_{k_2}-\omega_{k_3}-\omega_k\\ &+4\sum_l\,(T_{k_1 l}+T_{k_2 l}-T_{k_3 l}-T_{kl})\,n_l\\ &+4\,(T_{k_3}+T_k+T_{k_1 k_2}+T_{k_3 k}\\ &-T_{k_1 k}-T_{k_2 k}-T_{k_1 k_3}-T_{k_2 k_3})\Big)\ .\end{aligned} \tag{20}$$

It is obtained by moving the τ operator dependence entirely to either the left or the right, where it becomes a normal function of τ after acting on the states, and then using the representation $\int_{-\infty}^{+\infty} e^{i\tau x}\,d\tau = 2\pi\,\delta(x)$. This rather asymmetric looking expression can be recast as,

$$\begin{aligned}\delta\Big(&(\omega_{k_1}-2T_{k_1})+(\omega_{k_2}-2T_{k_2})\\ &-(\omega_{k_3}-2T_{k_3})-(\omega_k-2T_k)\\ &+2\sum_l\,(T_{k_1 l}+T_{k_2 l}-T_{k_3 l}-T_{kl})\\ &\times(2\,n_l+\delta_{k_1 l}+\delta_{k_2 l}-\delta_{k_3 l}-\delta_{kl})\Big).\end{aligned} \tag{21}$$

If we define an effective energy per state by

$$\begin{aligned}\varepsilon_m = \omega_m - 2\,T_m + 2\sum_l\,T_{ml}\ (\ 2\,n_l+\delta_{k_1 l}\\ +\delta_{k_2 l}-\delta_{k_3 l}-\delta_{kl}),\end{aligned} \tag{22}$$

then the δ function can be written most simply as

$$\delta(\ \varepsilon_{k_1}+\varepsilon_{k_2}-\varepsilon_{k_3}-\varepsilon_k\ )\ . \tag{23}$$

The exact expression for ε_m given above is greatly simplified in the large-n_l situation in which we work, so that

$$\varepsilon_m \approx \omega_m + 4\sum_l\,T_{ml}\,n_l. \tag{24}$$

Let us emphasize that the leading-order terms that are cubic in the mode numbers in (18) do not fully agree with the expressions found in [1, 2]; our leading-order terms can be written as

$$\begin{aligned}\left\langle \frac{d}{dt}\hat{n}_k\right\rangle = 8\,\pi \sum_{k_1,k_2,k_3} &|T'_{k_1 k_2 k_3 k}|^2\ \delta(\ \varepsilon_{k_1}+\varepsilon_{k_2}-\varepsilon_{k_3}-\varepsilon_k\ )n_{k_1}\,n_{k_2}\,n_{k_3}\,n_k\\ &\left[4\left(\frac{1}{n_{k_3}}+\frac{1}{n_k}-\frac{1}{n_{k_1}}-\frac{1}{n_{k_2}}\right)\right.\\ &\left.-2\,\delta_{k_1 k_2}\left(\frac{1}{n_{k_3}}+\frac{1}{n_k}\right)+2\,\delta_{k_3 k}\left(\frac{1}{n_{k_1}}+\frac{1}{n_{k_2}}\right)\right],\end{aligned} \tag{25}$$

with ε_m given by Eq. (24). One clear difference in our result concerns the last two terms in (25), which are not present in [1, 2]; these would still be there in our analysis even if we had perturbed around the quadratic term in H. In fact, further diagonal terms would presumeably be present as well, but these we have accounted for by incorporating them into H_0. This discrepancy will be discussed in the following section. The other clear difference is due to our different perturbation expansion, which results in the δ function involving the effective energy ε_k rather than ω_k.

III. STATIONARY SOLUTIONS

The analysis of stationary solutions to the kinetic equation in the usual perturbative version can be found in [1, 2]. In that analysis, the extra terms we found in (25) are not considered and they analyze the condition

$$0 = \sum_{k_1,k_2,k_3} |T_{k_1 k_2 k_3 k}|^2 \, \delta(\omega_{k_1} + \omega_{k_2} - \omega_{k_3} - \omega_k) \quad (26)$$
$$\times n_{k_1} \, n_{k_2} \, n_{k_3} \, n_k \left\{ \frac{1}{n_{k_3}} + \frac{1}{n_k} - \frac{1}{n_{k_1}} - \frac{1}{n_{k_2}} \right\} .$$

The question therefore arises as to whether the other terms in Eq. (25), which are also cubic in the mode numbers, are small relative to the others. In this regard, it is noteworthy that the additional terms involve one less sum over momentum space. So if the terms in the summation are large over some reasonable domain in momentum space, then those factors will be suppressed by roughly the volume of that domain. Our calculation thus far has assumed that at least some of the mode numbers are large compared to unity and neglecting these additional two terms amounts to some kind of additional condition such as the one just suggested. One might instead entertain a more restricted class of the $T'_{k_1 k_2 k_3 k_4}$ coefficient such that the additional terms vanish identically; perhaps this might be natural in the context of vortex dynamics. For example, it could contain the factor $|k_1 - k_2|^\sigma |k_3 - k_4|^\sigma$ for $\sigma > 0$. We will not consider this issue further here and will proceed to consider the stationary solutions to Eq. (25), assuming that the last two terms can be neglected.

Following the analysis in the usual perturbative expansion [1, 2], one solution for the occupation numbers n_m is given by

$$n_m = \frac{T}{\mu + \varepsilon_m} \quad , \quad (27)$$

where T and μ are constants; this is the large-T limit of the usual thermodynamic distribution $\{\exp[(\varepsilon_k + \mu)/T] - 1)\}^{-1}$ [3] for noninteracting bosons. The difference in our case is simply that the effective energy ε_m enters rather than ω_m. Given the precise form of this effective energy, we see that Eq. (27) is in fact a self-contained integral equation for n_m, but we will not analyze it further here.

While the preceding analysis can be carried out equally well in terms of discrete momenta and sums (which we have done) and continuous variables with integrals, the examination of the Kolmogorov solutions requires the later setting. The only care in going over to integral expressions is in correctly treating δ-function factors. It is generally the case that the perturbative expansion we have considered will contain a factor of $\delta(0)$, which must be factored out and discarded, and it is therefore convenient now to take the $T_{k_1 k_2 k_3 k_4}$ coefficient without the momentum conserving δ function $\delta^{(d)}(k_1 + k_2 - k_3 - k)$ we previously included so that the integral equivalent of Eq. (25) has only a single momentum-conserving δ function.

One way to establish the Kolmogorov solutions is to follow the presentation in [1, 2], substituting the effective energy ε_m for the free field energy ω_m. This entails a number of assumptions. We will assume that the theory has rotational symmetry, which implies, in particular, that the mode number $n_k = n(k)$ and the effective energy $\varepsilon_k = \varepsilon(k)$ depend only on the magnitude of the vector k. We further suppose that the scaling properties $\varepsilon(\lambda k) = \lambda^\alpha \varepsilon(k)$ and $T'(\lambda k_1, \lambda k_2, \lambda k_3, \lambda k_4) = \lambda^\beta T'(k_1, k_2, k_3, k_4)$ are satisfied and that the functional relation $\varepsilon(k)$ is invertible. Without loss of generality, we can take $\varepsilon(0) = 0$ by adjusting, say, the $\omega(0)$ coefficient if necessary. Given these assumptions, we will show that a solution exists of the form $n(k) \equiv n(\varepsilon) = \varepsilon^{-x}$ (repeated use of the symbol n for two different functions should cause no confusion; we will always regard the mode numbers as functions of the effective energy in the following).

Let us begin by integrating over angles in (25); the volume element is $d^d k_i = k_i^{d-1} \, dk_i \, d\Omega_i$ and it should be clear from the context whether we use the symbol k_i to denote a vector or its magnitude. Using the assumption that we can invert the relationship $\varepsilon(k)$, we can change variables in the remaining integrals from k_i to ε_i. It is convenient to first define

$$U(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon) = (k_1 \, k_2 \, k_3 \, k)^{d-1} \left| \frac{d\varepsilon_1}{dk_1} \frac{d\varepsilon_2}{dk_2} \frac{d\varepsilon_3}{dk_3} \frac{d\varepsilon}{dk} \right|^{-1} \quad (28)$$
$$\times \int |T'_{k_1 k_2 k_3 k}|^2$$
$$\times \delta^{(d)}(k_1 + k_2 - k_3 - k) \, d\Omega_1 \, d\Omega_2 \, d\Omega_3 \quad ,$$

where we are integrating over three sets of angular variables. Let us note that U shares the same symmetry properties as the coefficient T' under permutations of its arguments. Our task now is to find solutions to

$$0 = \int_0^\infty d\varepsilon_1 \, d\varepsilon_2 \, d\varepsilon_3 \, U(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon) \, \delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_3 - \varepsilon)$$
$$\times n_{k_1} \, n_{k_2} \, n_{k_3} \, n_k \left\{ \frac{1}{n_{k_3}} + \frac{1}{n_k} - \frac{1}{n_{k_1}} - \frac{1}{n_{k_2}} \right\} . \quad (29)$$

It is straightforward to work out the scaling properties of U that are implied by the assumptions. By simply scaling each of the momenta on both sides of the defining equation, one quickly finds the relation

$$U(\lambda \varepsilon_1, \lambda \varepsilon_2, \lambda \varepsilon_3, \lambda \varepsilon) = \lambda^\gamma \, U(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon) \quad , \quad (30)$$
$$\gamma = \frac{3d + 2\beta}{\alpha} - 4 \quad .$$

Returning to the analysis of Eq. (29), one easily uses

the δ function to carry out the ε_3 integral; what remains is a region D of the $(\varepsilon_1, \varepsilon_2)$ plane. This region is not the entire first quadrant since we must satisfy the condition

$$\varepsilon_3 = \varepsilon_1 + \varepsilon_2 - \varepsilon \geq 0 \ . \tag{31}$$

Divide D into four sectors as follows:

$$\begin{aligned} D_1 &= \{(\varepsilon_1, \varepsilon_2) \in D \mid \varepsilon_1 < \varepsilon, \quad \varepsilon_2 < \varepsilon\} \ , \\ D_2 &= \{(\varepsilon_1, \varepsilon_2) \in D \mid \varepsilon_1 > \varepsilon, \quad \varepsilon_2 > \varepsilon\} \ , \\ D_3 &= \{(\varepsilon_1, \varepsilon_2) \in D \mid \varepsilon_1 < \varepsilon, \quad \varepsilon_2 > \varepsilon\} \ , \\ D_4 &= \{(\varepsilon_1, \varepsilon_2) \in D \mid \varepsilon_1 > \varepsilon, \quad \varepsilon_2 < \varepsilon\} \end{aligned} \tag{32}$$

and perform the Zakharov transformations [2, 4] to map D_2, D_3, and D_4 onto D_1. Those transformations D_2, D_3, and D_4, respectively take the form

$$\begin{aligned} \varepsilon_1 &= \frac{\varepsilon\varepsilon_1'}{\varepsilon_1' + \varepsilon_2' - \varepsilon}, \quad \varepsilon_2 = \frac{\varepsilon\varepsilon_2'}{\varepsilon_1' + \varepsilon_2' - \varepsilon} \ ; \\ \varepsilon_1 &= \frac{\varepsilon(\varepsilon_1' + \varepsilon_2' - \varepsilon)}{\varepsilon_2'}, \quad \varepsilon_2 = \frac{\varepsilon^2}{\varepsilon_2'} \ ; \\ \varepsilon_1 &= \frac{\varepsilon^2}{\varepsilon_1'}, \quad \varepsilon_2 = \frac{\varepsilon(\varepsilon_1' + \varepsilon_2' - \varepsilon)}{\varepsilon_1'} \ . \end{aligned} \tag{33}$$

Using these transformations and the ansatz $n(\varepsilon) = C\,\varepsilon^{-x}$, one finds that Eq. (29) becomes

$$\begin{aligned} 0 = \int_{D_1} d\varepsilon_1\, d\varepsilon_2\, U(\varepsilon_1, \varepsilon_2, \varepsilon_1 + \varepsilon_2 - \varepsilon, \varepsilon)\, [\varepsilon_1\, \varepsilon_2\, (\varepsilon_1 + \varepsilon_2 - \varepsilon)\, \varepsilon]^{-x} \\ \times \{\, (\varepsilon_1 + \varepsilon_2 - \varepsilon)^x + \varepsilon^x - \varepsilon_1^x - \varepsilon_2^x \,\} \left\{ 1 + \left(\frac{\varepsilon_1 + \varepsilon_2 - \varepsilon}{\varepsilon}\right)^y - \left(\frac{\varepsilon_2}{\varepsilon}\right)^y - \left(\frac{\varepsilon_1}{\varepsilon}\right)^y \right\} \ , \end{aligned} \tag{34}$$

with $y = 3x - \gamma - 3$. We can satisfy this relation at the four points $x = 0$, $x = 1$, $y = 0$, or $y = 1$. The first two cases are simply limits of the thermodynamic distribution previously considered, while the last two possibilities correspond to the Kolmogorov solutions

$$n^{(1)}(\varepsilon) = C_1\, \varepsilon^{-(\gamma+3)/3} \quad , \quad n^{(2)}(\varepsilon) = C_2\, \varepsilon^{-(\gamma+4)/3} \ . \tag{35}$$

Although the form of these solutions is the same as in the usual perturbative analysis, they differ fundamentally in that they are in fact integral equations for $n(\varepsilon)$, owing to Eq. (24); we will not pursue an analysis of them here.

IV. CONCLUDING REMARKS

In this paper we have established a different perturbative expansion that leads to a quantum kinetic equation, similar to the usual one obtained by expanding about a quadratic Hamiltonian. By placing the diagonal quartic terms into the unperturbed Hamiltonian, we have generalized that equation and identified the effective state energy, which essentially takes the place of the free field energy in those earlier treatments. The framework we establish applies as well to the usual perturbative situation and can be considered as an alternative to other derivations. We have also highlighted some subtleties that arise there in the leading-order terms cubic in the mode numbers and have offered an argument why one might expect those additional corrections to be small.

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